SOLUTIONS FOR ADMISSIONS TEST IN
MATHEMATICS, COMPUTER SCIENCE AND JOINT SCHOOLS
WEDNESDAY 2 NOVEMBER 2011

Mark Scheme:

Each part of Question 1 is worth four marks which are awarded solely for the correct answer.

Each of Questions 2-7 is worth 15 marks

QUESTION 1:

A. Note that $y = x^3 - x^2 - x + 1 = (x-1)(x^2-1) = (x-1)^2(x+1)$ so that the cubic has a repeated root at 1 and a simple root at -1. The **answer is (c)**.

B. Let the rectangle have sides x, y so that

$$2x + 2y = P, \qquad xy = A.$$

Eliminating y we see that

$$x^2 - \frac{1}{2}Px + A = 0.$$

As x is real it follows that the discriminant is non-negative and so

$$\left(\frac{P}{2}\right)^2 - 4 \times 1 \times A \geqslant 0 \implies P^2 \geqslant 16A.$$

The **answer is (c)**. (This condition is in fact sufficient for x and y to be real and positive.)

C. We have $x_n = n^3 - 9n^2 + 631$ so that $x_n > x_{n+1}$ is equivalent to

$$\begin{aligned} & (n^3 - 9n^2 + 631) - ((n+1)^3 - 9(n+1)^2 + 631) \\ = \; & (n^3 - n^3 - 3n^2 - 3n - 1) - 9(n^2 - n^2 - 2n - 1) \\ = \; & -3n^2 + 15n + 8. \end{aligned}$$

We note

$$-3x^2+15x+8 = 0 \iff x = \frac{-15 \pm \sqrt{225+96}}{-6} \iff x = \alpha = \frac{15-\sqrt{321}}{6} \text{ or } x = \beta = \frac{15+\sqrt{321}}{6}.$$

So $-3x^2 + 15x + 8 > 0$ if and only if $\alpha < x < \beta$. As $18^2 = 324$ then $\beta \approx (15+18)/6 = 5\frac{1}{2}$ and so $n = 5$ is the largest integer in the range $\alpha < n < \beta$. The **answer is (a).**

D. In the range $0 \leqslant x \leqslant 2\pi$ we have

$$\sin x \geqslant \frac{1}{2} \text{ when } \frac{\pi}{6} \leqslant x \leqslant \frac{5\pi}{6}; \qquad \sin 2x \geqslant \frac{1}{2} \text{ when } \frac{\pi}{12} \leqslant x \leqslant \frac{5\pi}{12} \text{ and when } \frac{13\pi}{12} \leqslant x \leqslant \frac{17\pi}{12}.$$

One or both of these inequalities hold for x in the ranges

$$\frac{\pi}{12} \leqslant x \leqslant \frac{5\pi}{6} \text{ and } \frac{13\pi}{12} \leqslant x \leqslant \frac{17\pi}{12}$$

which are intervals of total length $13\pi/12$. The relevant fraction is $(13\pi/12)/(2\pi) = 13/24$. The **answer is (b).**

E. Without any loss of generality, we can assume the radius of the circle to be 1. Let A be the vertex of the angles α and β and B be the vertex of the angle γ. We see that AC has length $1/\sin\alpha$. Applying the sine rule to the triangle ABC we find

$$\frac{\sin\beta}{1} = \frac{\sin\gamma}{1/\sin\alpha}$$

and the **answer is (b).**

F. Note that

$$x^2 + y^2 + 4x\cos\theta + 8y\sin\theta + 10 = 0$$

rearranges to

$$(x + 2\cos\theta)^2 + (y + 4\sin\theta)^2 = 4\cos^2\theta + 16\sin^2\theta - 10 = 12\sin^2\theta - 6$$

and so the equation defines a circle of radius $\sqrt{12\sin^2\theta - 6}$ provided $\sin^2\theta > \frac{1}{2}$. In the given range, $0 \leqslant \theta < \pi$, we have $\sin\theta \geqslant 0$ and so we need $\frac{1}{\sqrt{2}} < \sin\theta$. **The answer is (b).**

G. For x in the interval $-1 \leqslant x \leqslant 1$, we have $-1 \leqslant x^2 - 1 \leqslant 0$. For t in the range $-1 \leqslant t \leqslant 0$ we can see from the graph that $f(t) = t + 1$. So for $-1 \leqslant x \leqslant 1$ we have $f(x^2 - 1) = (x^2 - 1) + 1 = x^2$ and hence

$$\int_{-1}^{1} f(x^2 - 1)\,\mathrm{d}x = \int_{-1}^{1} x^2\,\mathrm{d}x = \left[\frac{x^3}{3}\right]_{-1}^{1} = \frac{2}{3}$$

and the **answer is (d).**

H. Note that $\log_{0.5} 0.25 = 2$ and that if $x > 0$ then

$$8^{\log_2 x} = 2^{3\log_2 x} = \left(2^{\log_2 x}\right)^3 = x^3$$

and likewise $9^{\log_3 x} = x^2 = 4^{\log_2 x}$. The given equation then rearranges to

$$\begin{aligned} & x = x^3 - x^2 - x^2 + 2 \\ \iff\quad & x^3 - 2x^2 - x + 2 = 0 \\ \iff\quad & (x-2)(x^2 - 1) = 0 \\ \iff\quad & (x-2)(x-1)(x+1) = 0. \end{aligned}$$

So the positive roots are $x = 1, 2$ and the **answer is (c)**.

I. We know, for $0 \leqslant x < 2\pi$, that $\sin^2 x + \cos^2 x = 1$ and that $0 \leqslant \sin^2 x, \cos^2 x \leqslant 1$. Also, for any y in the range $0 \leqslant y \leqslant 1$, then $y^3 < y$ and $y^4 < y$ unless $y = 0$ or 1. So we would have

$$\sin^8 x + \cos^6 x < \sin^2 x + \cos^2 x = 1$$

unless $\sin^2 x$ and $\cos^2 x$ are 0 and 1 in some order. So the only solutions are $x = 0, \pi/2, \pi, 3\pi/2$ in the given range and the **answer is (b).**

Alternatively, and writing $s = \sin\theta$ and $c = \cos\theta$ to ease the notation, we see

$$\begin{aligned} & s^8 + c^6 = 1 \\ \iff\quad & s^8 + (1 - s^2)^3 = 1 \\ \iff\quad & s^8 - s^6 + 3s^4 - 3s^2 = 0 \\ \iff\quad & s^2(s^6 - s^4 + 3s^2 - 3) = 0 \\ \iff\quad & s^2(s^2 - 1)(s^4 + 3) = 0 \end{aligned}$$

to see that we have roots when $s = 0, 1, -1$ at $x = 0, \pi, \pi/2, 3\pi/2$ as before.

J. We note

$$\begin{aligned} f(1) &= 1, \qquad f(2) = f(1) = 1, \qquad f(3) = f(1)^2 - 2 = -1, \qquad f(4) = f(2) = 1, \\ f(5) &= f(2)^2 - 2 = -1, \qquad f(6) = f(3) = -1, \qquad f(7) = f(3)^2 - 2 = -1, \qquad f(8) = f(4) = 1. \end{aligned}$$

The function f can only take values 1 and -1 because of the nature of the two rules defining f; specifically if we set $f(n) = \pm 1$ into either of the two rules we can only achieve ± 1 for further values of f. Moreover $f(n) = 1$ is only possible when the first rule $f(2n) = f(n)$ has been repeatedly applied to determine $f(n)$; any use of the second rule would lead to $f(n) = -1$. Repeated application of only the first rule will determine $f(n)$ when n is a power of 2; for such n we have $f(n) = 1$ and otherwise $f(n) = -1$.

Amongst the numbers $1 \leqslant n \leqslant 100$ the powers of 2 are $1, 2, 4, 8, 16, 32, 64$. So there are 7 powers of 2 and 93 other numbers. Hence

$$f(1) + f(2) + f(3) + \cdots + f(100) = 1 \times 7 + (-1) \times 93 = -86$$

and the **answer is (a).**

2. (i) [2 marks] Multiplying $x^3 = 2x + 1$ by x we get the first equation. Multiplying by x again we get

$$x^5 = x^2 + 2x^3 = x^2 + 2\left(2x + 1\right) = 2 + 4x + x^2.$$

(ii) [5 marks] If $x^k = A_k + B_k x + C_k x^2$ then

$$\begin{aligned} x^{k+1} &= A_k x + B_k x^2 + C_k x^3 \\ &= A_k x + B_k x^2 + C_k \left(2x + 1\right) \\ &= C_k + \left(A_k + 2C_k\right) x + B_k x^2. \end{aligned}$$

Comparing the coefficients of $1, x, x^2$ we then get

$$A_{k+1} = C_k, \qquad B_{k+1} = A_k + 2C_k, \qquad C_{k+1} = B_k.$$

(iii) [5 marks] With $D_k = A_k + C_k - B_k$ we have

$$\begin{aligned} D_{k+1} &= A_{k+1} + C_{k+1} - B_{k+1} \\ &= C_k + B_k - \left(A_k + 2C_k\right) \qquad \text{[using the previous part]} \\ &= B_k - A_k - C_k \\ &= -D_k. \end{aligned}$$

As $1 = 1 + 0x + 0x^2$ then $A_0 = 1$, $B_0 = 0$, $C_0 = 0$ and $D_0 = 1$. So $D_k = \left(-1\right)^k$ and

$$A_k + C_k = B_k + \left(-1\right)^k.$$

(iv) [3 marks] Let $F_k = A_{k+1} + C_{k+1}$. Then

$$\begin{aligned} F_k + F_{k+1} &= \left(A_{k+1} + C_{k+1}\right) + \left(A_{k+2} + C_{k+2}\right) \\ &= B_{k+1} + \left(-1\right)^{k+1} + B_{k+2} + \left(-1\right)^{k+2} \\ &= B_{k+1} + B_{k+2} \\ &= C_{k+2} + C_{k+3} \\ &= A_{k+3} + C_{k+3} \\ &= F_{k+2}. \end{aligned}$$

3. (i) [2 marks] As $y = m(x-a)$ and $y = x^3 - x$ touch at $x = b$ then their gradients agree and so, differentiating, $m = 3b^2 - 1$.

(ii) [3 marks] As they also meet when $x = b$, then the y-coordinates of the graphs also agree so that

$$b^3 - b = m(b-a) = \left(3b^2 - 1\right)(b-a).$$

Solving for a we have

$$a = b - \left(\frac{b^3 - b}{3b^2 - 1}\right) = \frac{3b^3 - b - b^3 + b}{3b^2 - 1} = \frac{2b^3}{3b^2 - 1}.$$

(iii) [3 marks] If a is a large negative number then the line $y = m(x-a)$ will be almost horizontal meaning that the line will be tangential very close to the cubic's (local) maximum where $x = -1/\sqrt{3}$. Alternatively one could argue that $a = 2b^3/\left(3b^2 - 1\right) \ll 0$ only when $3b^2 - 1 \approx 0$ and so $b \approx -1/\sqrt{3}$ as $b < 0$.

(iv) [2 marks] If we expand $(x-b)^2(x-c)$ and compare the coefficients of x^2 we get $0 = -2b - c$ and so $c = -2b$.

(v) [5 marks] We can see that as a increases then the tangent line rises and so the area of R increases. So this area is greatest when $a = -1$. In this case $b = a = -1$ and $c = 2$. Hence the largest area achieved by R is

$$\begin{aligned}
\int_{-1}^{2} \left[2(x+1) - \left(x^3 - x\right)\right] \mathrm{d}x &= \int_{-1}^{2} \left[3x + 2 - x^3\right] \mathrm{d}x \\
&= \left[\frac{3x^2}{2} + 2x - \frac{x^4}{4}\right]_{-1}^{2} \\
&= (6 + 4 - 4) - \left(\frac{3}{2} - 2 - \frac{1}{4}\right) = \frac{27}{4}.
\end{aligned}$$

4.

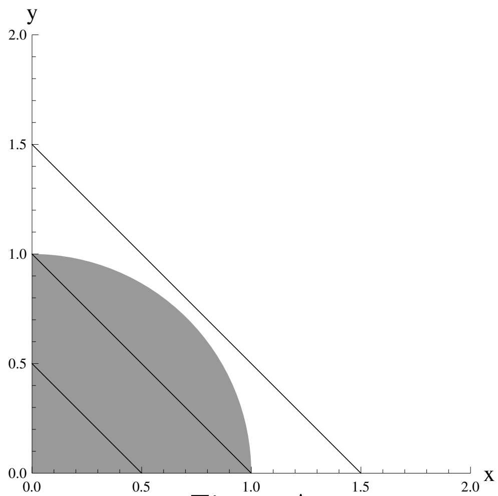

Figure A

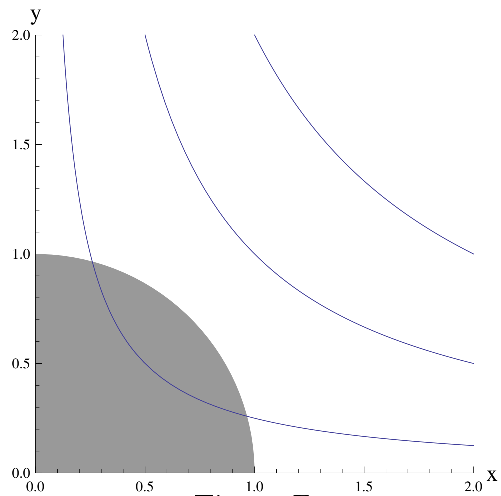

Figure B

(i) [4 marks] The largest value of $x + y$ will be achieved at the point where $x + y = k$ is tangential to the boundary of Q. By symmetry, this occurs at $\left(1/\sqrt{2}, 1/\sqrt{2}\right)$. So the largest value of $x + y$ on Q is $\sqrt{2}$.

(ii) [6 marks] Again we can see that the maximum value of xy on Q is when $xy = k$ is tangential to Q's boundary. This takes place once more at $\left(1/\sqrt{2}, 1/\sqrt{2}\right)$ where $xy = 1/2$. As $x^2 + y^2$ also takes its largest value (of 1) there then the maximum of

$$x^2 + y^2 + 4xy = 1 + \frac{4}{2} = 3.$$

Further xy takes a minimum value of 0 on Q (on the axes). At $(1, 0)$ and $(0, 1)$ we also have $x^2 + y^2$ taking its maximum value (of 1) and so we have

$$x^2 + y^2 - 6xy = 1 - 6 \times 0 = 1$$

as the largest value of $x^2 + y^2 - 6xy$ achieved at points (x, y) in Q.

(iii) [5 marks] As $x^2 + y^2 - 4x - 2y = k$ can have the squares completed to become

$$(x - 2)^2 + (y - 1)^2 = k + 5$$

we see that the curve is the circle with centre $(2, 1)$ and radius $\sqrt{k + 5}$.

So the function $x^2 + y^2 - 4x - 2y$ increases as we move away from $(2, 1)$ and the closest point to $(2, 1)$ in Q is $\frac{1}{\sqrt{5}}(2, 1)$. At that point

$$x^2 + y^2 - 4x - 2y = \left(\frac{2}{\sqrt{5}}\right)^2 + \left(\frac{1}{\sqrt{5}}\right)^2 - \frac{8}{\sqrt{5}} - \frac{2}{\sqrt{5}} = 1 - \frac{10}{\sqrt{5}} = 1 - 2\sqrt{5}$$

or we can note

$$k = \left(\sqrt{5} - 1\right)^2 - 5 = 1 - 2\sqrt{5}.$$

5. (i) [2 marks] All solutions in a semi-grid of size 4 are as follows:

$$RRR, \quad RRU, \quad RUR, \quad RUU, \quad URR, \quad URU, \quad UUR, \quad UUU.$$

(ii) [2 marks] Write R for a right-move and U for an up-move. A solution in a semi-grid of size n is then a sequence of $n-1$ letters, U or R.

(iii) [2 marks] Each solution consists of $n-1$ letters, and each letter can be chosen independently; thus there are 2^{n-1} solutions in total.

(iv) [3 marks] In a semi-grid of size 4 there are now ten solutions: R, U and the eight solutions from part (ii).

In a semi-grid of size 5: the same ten solutions.

(v) [6 marks] The modified definitions ensure that the goal squares are actually the squares on the diagonals of semi-grids of sizes $2, 4, 6, \ldots$ and so on, up to $n-1$ (if n is odd) or n (if n is even). Then, by using the result from part (iii) of this question, the total number of paths from the original location to a goal square can be obtained as follows. If $n = 2k$ then the sum is

$$2^{2-1} + 2^{4-1} + \cdots + 2^{2k-1} = \frac{1}{2}\sum_{i=1}^{k} 2^{2i}.$$

And in fact if $n = 2k+1$ then we arrive at the same sum.

As this is the sum of a geometric progression, we obtain

$$\frac{1}{2}\left(\frac{4\left(4^k-1\right)}{4-1}\right) = \frac{2}{3}\left(4^k-1\right).$$

This can also be written as $\frac{2}{3}\left(2^{2k}-1\right)$ or $\frac{2}{3}\left(2^{n-1}-1\right)$ when n is odd and $\frac{2}{3}\left(2^{n}-1\right)$ when n is even.

6. (i) [2 marks] Alice and Bob's statements are contradictory, so one of them must be lying. (Alternatively: one of Bob and Diane must be lying.)

(ii) [2 marks] If Bob is lying then Diane is telling the truth; hence at least one of them is telling the truth.

(iii) [3 marks] Either Alice or Bob is lying. If Alice is lying and Bob is telling the truth, then Diane is also lying – contradiction. Hence Bob is lying, Alice is telling the truth, so Bob broke the vase. Alternatively: either Bob or Diane is lying; hence Alice and Charlie are telling the truth. Therefore Bob broke the vase (Alice's statement) and Bob is the liar.

(iv) [3 marks] We know already that either Bob or Diane is telling the truth; hence Alice and Charlie are lying. Therefore it was Charlie who broke the vase, Bob is lying, and Diane is telling the truth.

(v) [4 marks] Either Bob or Diane is telling the truth.
(a) If Bob is telling the truth, then Diane broke the vase, Charlie is telling the truth, and Alice and Diane are lying.
(b) If Diane is telling the truth, then Bob is lying. If Alice is telling the truth then Charlie is also telling the truth – contradiction. So Alice is lying, Charlie is telling the truth. Hence it wasn't Bob, Charlie or Diane, so by elimination it was Alice.

(vi) [1 mark] The scenarios above include cases where each of the four broke the vase.

7. (i) [5 marks] Minimum 3: TTT or HHH are examples, and no shorter sequence contains any pair other than the one just completed. Maximum 6: an example of a non-losing 5-element sequence is HTTHH; any sequence with more than 5 elements contains at least 5 adjacent pairs, and there are only 4 distinct pairs.

(ii) [3 marks] As an example, Bob can win by duplicating Alice's first move and negating her second move (if she doesn't lose on her second move). The list of possible games is then HHH, HHTH*, TTHT*, TTT where * must be a losing move.

(iii) [2 marks] If Alice repeatedly plays H and Bob duplicates with H, then Bob loses on move $n+1$, having produced a second occurrence of n Hs.

(iv) [5 marks] In a non-losing sequence, some n-element sequence will begin at position 1; a different one will begin at position 2, and so on. There are 2^n distinct sequences of length n, thus no new n-element sequence could begin beyond 2^n. If the final distinct sequence begins at 2^n it ends at $2^n + n$, and any element that follows it must lose the game. Therefore no game of HT-n can involve more than $2^n + n$ moves (including the losing move). The question asks whether 120 moves is an upper bound, rather than a least upper bound, so calculating $2^6 + 6 = 70 < 120$ is sufficient.